In the name of GOD

TISSUE MECHANICS

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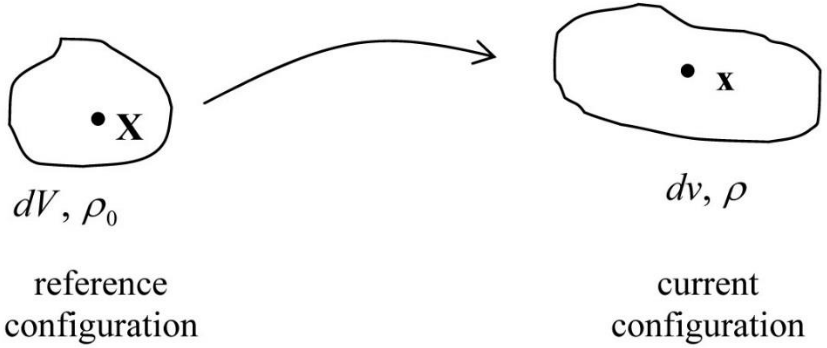
Mass

- **What is Mass?** A non-negative scalar measure of a body's tendency to resist a change in motion.

$$\rho_{\mathrm{AVE}} = \frac{\Delta m}{\Delta v} \longrightarrow \rho(\mathbf{x},t) = \lim_{\Delta v \to 0} \frac{\Delta m}{\Delta v} \longrightarrow \boxed{m = \int_v \rho(\mathbf{x},t)\,dv}$$

- **The law of Conservation of Mass:** Mass can neither be created nor destroyed.

$$dm = \rho_0(\mathbf{X})dV = \rho(\mathbf{x},t)dv$$



$$m = \int_V \rho_0(\mathbf{X})dV = \int_v \rho(\mathbf{x},t)dv = \mathrm{const}$$

$$\dot{m} = \frac{dm}{dt} = \frac{d}{dt}\int_v \rho(\mathbf{x},t)dv = 0$$

Continuity Equation

Eulerian: $\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0$ or $\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0$ $\xrightarrow{\rho = \mathrm{const}}$ $\mathrm{div}\,\mathbf{v} = 0, \quad \frac{\partial v_i}{\partial x_i} = 0$

Lagrangian: $\rho_0(\mathbf{X}) = \rho(\boldsymbol{\chi}(\mathbf{X},t),t)J(\mathbf{X},t)$ or $\frac{d}{dt}(\rho J) = 0$ $\xrightarrow{\rho = \mathrm{const}}$ $\mathrm{div}\,\mathbf{v} = 0, \quad \frac{\partial v_i}{\partial x_i} = 0$

Example:

The velocity of a particle is given as follows:

$$v_i = \frac{kx_i}{1 + kt}$$

Get the density of the particle as a function of time.

Solve:

According to the law of conservation of mass: $$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0$$

$$\frac{\partial \rho}{\partial t} = -\rho \frac{\partial v_i}{\partial x_i} = -\rho k \frac{\delta_{ij}}{1+kt} = -\frac{3\rho k}{1+kt} \rightarrow \int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = -\int_0^1 \frac{3kdt}{1+kt}$$

$$\rho = \frac{\rho_0}{(1 + kt)^3}$$

Momentum

The basic dynamics principles: Newton's Laws (force equilibrium and moment equilibrium)

- ✓ Newton's first low: In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.
- ✓ Newton's second low: In an inertial reference frame, the vector sum of the forces F on an object is equal to the mass *m* of that object multiplied by the acceleration a of the object: F = *m*a.
- ✓ Newton's third low: When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

An alternative but completely equivalent set of dynamics laws are Euler's Laws; these are:

- More appropriate for finite-sized collections of moving particles
- Can be used to express the force and moment equilibrium in terms of integrals
- Called the Momentum Principles:

➢ The principle of linear momentum (Euler's first law)

➢ The principle of angular momentum (Euler's second law).

The Principle of Linear Momentum

- **What is Momentum?** A measure of the tendency of an object to keep moving once it is set in motion.

- $P = mv \rightarrow$ $\frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$
- $F = ma$

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \xrightarrow{F=0} mv = const$$

The principle of linear momentum, or balance of linear momentum: *The rate of change of momentum is equal to the applied force*

The law of conservation of linear momentum

The Principle of Linear Momentum

In continuum mechanics:

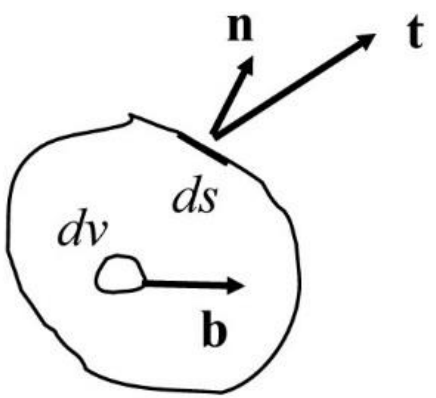
- Linear momentum → $\mathbf{L}(t) = \int_v \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)dv$
- The principle of linear momentum

$$\dot{\mathbf{L}}(t) = \frac{d}{dt}\int_v \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)dv = \mathbf{F}(t)$$

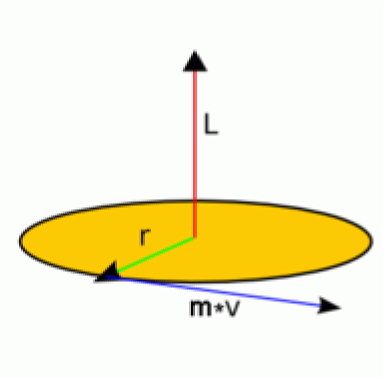
$$\mathbf{F}(t) = \int_s \mathbf{t}ds + \int_v \mathbf{b}dv, \qquad F_i = \int_s t_i ds + \int_v b_i dv$$

$$\int_s \mathbf{t}ds + \int_v \mathbf{b}dv = \int_v \rho\dot{\mathbf{v}}dv$$

Principle of Linear Momentum



The Principle of Angular Momentum



- **Angular momentum** is the rotational equivalent of linear momentum.
- The angular momentum **h:** $\mathbf{h} = \mathbf{r} \times m\mathbf{v}$
- **The principle of angular momentum:** the resultant moment of the external forces acting on the system of particles, **M**, equals the rate of change of the total angular momentum of the particles**:**

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \frac{d\mathbf{h}}{dt}$$

- In continuum mechanics: $\mathbf{H} = \int_v \mathbf{r} \times \rho \mathbf{v} dv$

$$\mathbf{F}(t) = \int_s \mathbf{t} ds + \int_v \mathbf{b} dv, \qquad F_i = \int_s t_i ds + \int_v b_i dv$$

$$\int_s \mathbf{r} \times \mathbf{t}^{(\mathbf{n})} ds + \int_v \mathbf{r} \times \mathbf{b} dv = \frac{d}{dt} \int_v \mathbf{r} \times \rho \mathbf{v} dv$$

Principle of Angular Momentum

The Equations of Motion

- Cauchy's law $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$
- The Principle of Linear Momentum
- The Principle of Angular Momentum

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\frac{d\mathbf{v}}{dt}, \qquad \frac{\partial\sigma_{ij}}{\partial x_j} + b_i = \rho\frac{dv_i}{dt}$$

Equations of Motion

acceleration = 0

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = 0$$

Equations of Equilibrium

Example

1) In the absence of body force, does the distribution of the following stress follows equilibrium equations?

$$\sigma_{11} = x_2^2 + v(x_1^2 - x_2^2) \quad ,\sigma_{12} = -2vx_1x_2 \quad ,\sigma_{22} = x_1^2 + v(x_2^2 - x_1^2)$$
$$\sigma_{23} = 0, \qquad \sigma_{13} = 0, \qquad \sigma_{33} = v(x_2^2 + x_1^2)$$

Solve:

According to the equations of equilibrium: $\mathrm{div}\,\boldsymbol{\sigma} + \mathbf{b} = 0$

$$\frac{\partial\sigma_{1j}}{\partial x_j} = \frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} = 2vx_1 - 2vx_1 + 0 = 0$$

$$\frac{\partial\sigma_{2j}}{\partial x_j} = \frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} = -2vx_2 + 2vx_2 + 0 = 0$$

$$\frac{\partial\sigma_{3j}}{\partial x_j} = \frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} = 0 + 0 + 0 = 0$$

2) If $\sigma_{ij} = -p\delta_{ij}$

Where $p = p(x_1, x_2, x_3, t)$

The **equations of motion given as:**

$$\frac{\partial\sigma_{ij}}{\partial x_j} + b_i = \rho\frac{dv_i}{dt}$$

$$\frac{\partial\sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ij} = -\frac{\partial p}{\partial x_i}$$

$$-\frac{\partial p}{\partial x_i} + b_i = \rho\frac{dv_i}{dt}$$

Balance of Mechanical Energy

- W: work, K: kinetic energy $\rightarrow$

$$W_{\text{ext}} + W_{\text{int}} = \Delta K \xrightarrow{\frac{d}{dt}} P_{\text{ext}} + P_{\text{int}} = \dot{K}$$

$$P_{\text{ext}} = \frac{d}{dt} W_{\text{ext}}, \quad P_{\text{int}} = \frac{d}{dt} W_{\text{int}}, \quad \dot{K} = \frac{d}{dt} \Delta K$$

$$\mathbf{F}_{\text{ext}} = \int_s \mathbf{t}\, ds + \int_v \mathbf{b}\, dv \xrightarrow{P = Fv} P_{\text{ext}} = \int_s \mathbf{t} \cdot \mathbf{v}\, ds + \int_v \mathbf{b} \cdot \mathbf{v}\, dv$$

$$K = \int_v \tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} dv \longrightarrow \frac{d}{dt} K = \int_v \tfrac{1}{2} \rho \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) dv = \int_v \rho \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} dv$$

$$\int_s \mathbf{t} \cdot \mathbf{v}\, ds + \int_v \mathbf{b} \cdot \mathbf{v}\, dv + P_{\text{int}} = \int_v \rho \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} dv$$

power of surface forces; power of body forces; power of internal forces; rate of change of kinetic energy

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{b} = 0$$

$$P_{\text{int}} = -\int_v \sigma_{ij} \left\{ \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right\} dv \longrightarrow P_{\text{int}} = \frac{d}{dt} U = \int_v \rho \frac{du}{dt} dv$$

Stress Power

Mechanical Energy Balance

$$\dot{K} + \dot{U} = P$$

$$Q = \int_V \rho r dV - \int_S q_i n_i dS$$

$$\dot{K} + \dot{U} = P + Q, \qquad \rho \dot{u} - \sigma_{ij} D_{ij} - \rho r + q_{i,i} = 0$$

Energy Equation

Entropy inequality

$$\rho \frac{D\eta}{Dt} \geq -div\left(\frac{q}{\theta}\right) + \frac{\rho q_s}{\theta}$$

- θ: absolute temperature
- η: entropy
- q: thermal flux vector
- q_s: internal energy

Entropy increase rate in a particle ≥ Entropy enters from the surface boundaries + The total internal entropy of the total volume

Boundary Conditions and The Boundary Value Problem

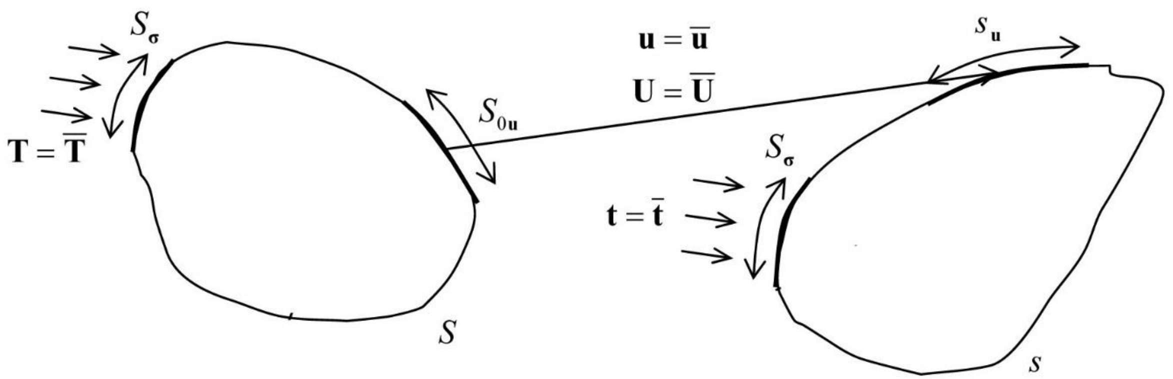
- In order to solve a mechanics problem, one must specify certain conditions around the boundary of the material under consideration.

Initial Conditions

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \mathbf{u}_0(\mathbf{x}), \quad \text{at } t = 0$$
$$\dot{\mathbf{u}}(\mathbf{x},\mathbf{t}) = \dot{\mathbf{u}}_0(\mathbf{x}), \quad \text{at } t = 0$$

Boundary Conditions

- **Displacement Boundary Conditions** $\quad \mathbf{u} = \overline{\mathbf{u}}, \quad \text{on } s_u$
- **Traction Boundary Conditions** $\quad \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \overline{\mathbf{t}}, \quad \text{on } s_\sigma$



Boundary value problem (BVP)

Unknown:

- Displacement (in 3 directions)
- Stress (6 Components)
- Strain (6 Components)

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Equations:

- Equilibrium Equations (3 equations): $\operatorname{div} \boldsymbol{\sigma} + \mathbf{b} = 0$

- constitutive equation (Hooke's Law) → relationship between stress and strain

$$\boldsymbol{E} = \frac{1+v}{E_y}\left(\boldsymbol{\sigma} - \frac{v}{1+v}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) I\right)$$

Modulus of elasticity

Poisson coefficient

- Compatibility equations (6 equations) → relationship between strain and displacement

$$\boldsymbol{E} = \frac{1}{2}(\nabla \boldsymbol{u} + \boldsymbol{u} \nabla)$$